CSE 332: Data Structures and Parallelism

Section 3: Recurrences and Closed Forms

Terminology	Recurrence Function/Relation	General formula	Closed form
Definition	Piecewise function that mathematically models the runtime of a recursive algorithm (might want to define constants)	Function written as the number of expansion i and recurrence function (might have a summation)	General formula evaluated without recurrence function or summations (force them to be in terms of constants or n)
Example	$T(n) = c_1$, for $n = 1$ $T(n) = T\left(\frac{n}{2}\right) + c_2$, otherwise	$T(n) = T\left(\frac{n}{2^i}\right) + i \cdot c_2$	Let $i = \log_2 n$, $T(n) = T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \cdot c_2$ $= T(1) + \log_2 n \cdot c_2$ $= c_1 + \log_2 n \cdot c_2$

0. Not to Tree

Consider the function $f(n)$. Find a recurrence modeling the worst-case runtime of this function and then find a Big-Oh bound for this recurrence.

```
f(n) {
    if (n <= 0) {
        return 1;
    }
    return 2 * f(n - 1) + 1;
}
```

a) Find a recurrence $T(n)$ modeling the *worst-case runtime complexity* of $f(n)$

> $T(n) = c_0$, if $n \le 0$
> $T(n) = T(n - 1) + c_1$, otherwise

b) Use your answer in part (a) to find a closed form for $T(n)$

> Unrolling the recurrence, we get
> $T(n) = T(n - 1) + c_1$
> $= T(n - 2) + c_1 + c_1$
> $= T(0) + c_1 + \dots + c_1$
> $= c_0 + c_1 + \dots + c_1$
> $= c_0 + n \cdot c_1$

1. To Tree

Consider the function $h(n)$. Find a recurrence modeling the worst-case runtime of this function and then find a Big-Oh bound for this recurrence.

```
h(n) {
    if (n <= 1) {
        return 1
    } else {
        return h(n/2) + n + 2*h(n/2)
    }
}
```

a) Find a recurrence $T(n)$ modeling the *worst-case runtime complexity* of $h(n)$

$T(n) = c_0$, if $n \leq 1$

$T(n) = 2T\left(\frac{n}{2}\right) + c_1$, otherwise

b) Use your answer in part (a) to find a closed form for $T(n)$

i	Recursive Call	# Nodes	Tree	Sum Work
0	$T(n)$	1	c_1	c_1
1	$T\left(\frac{n}{2}\right)$	2	c_1 + c_1	$2c_1$
2	$T\left(\frac{n}{2^2}\right)$	2^2	c_1 + c_1 + c_1 + c_1	2^2c_1
$\vdots$	$\vdots$	$\vdots$		$\vdots$
i	$T\left(\frac{n}{2^i}\right)$	2^i	c_1 + c_1 + ... + c_1 + c_1	2^ic_1
Base case occurs when $\frac{n}{2^i} = 1$ (i.e. $i = \log_2 n$)				
$\log n$	$T(1)$	n	c_0 + c_0 + ... + c_0 + c_0	$2^{\log n}c_0$

The recursion tree has height $\lg(n)$, each non-leaf level i has work $c_1 2^i$, and the leaf level has work $c_0 2^{\lg(n)}$. Putting this together, we have:

$$\left(\sum_{i=0}^{\lg(n)-1} c_1 2^i\right) + c_0 2^{\lg(n)} = c_1\left(\sum_{i=0}^{\lg(n)-1} 2^i\right) + c_0 n$$

$$= c_1 \frac{1-2^{\lg(n)}}{1-2} + c_0 n$$

$$= c_1\left(2^{\lg(n)} - 1\right) + c_0 n$$

$$= c_1(n - 1) + c_0 n$$

$$= \left(c_0 + c_1\right)n - c_1$$

2. To Tree or Not to Tree

Consider the function $f(n)$. Find a recurrence modeling the worst-case runtime of this function and then find a Big-Oh bound for this recurrence.

```
f(n) {
    if (n <= 1) {
        return 0
    }
    int result = f(n/2)
    for (int i = 0; i < n; i++) {
        result *= 4
    }
    return result + f(n/2)
}
```

a) Find a recurrence $T(n)$ modeling the *worst-case runtime complexity* of $f(n)$

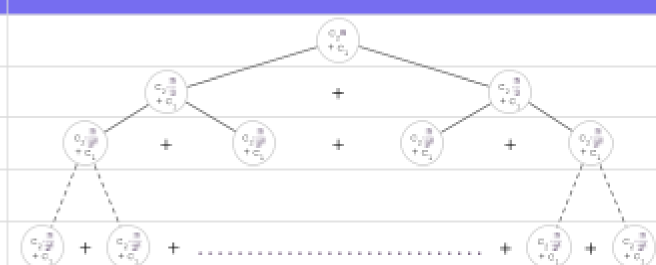
We look at the three separate components (base case, non-recursive work, recursive work). The base case is a constant amount of work, because we only do a return statement. We'll label it c_0. The non-recursive work is a constant amount of work (we'll call it c_1) for the assignments and `if` tests and a constant (we'll call c_2) multiple of n for the loops. The recursive work is $2T\left(\frac{n}{2}\right)$.

Putting these together, we get:

$T(n) = c_0$, if 1

$T(n) = 2T\left(\frac{n}{2}\right) + c_2 n + c_1$, otherwise

b) Use your answer in part (a) to find a closed form for $T(n)$

i	Recursive Call	# Nodes	Tree	Sum Work
0	$T(n)$	1		$c_2 n + c_1$
1	$T\left(\frac{n}{2}\right)$	2		$c_2 n + 2c_1$
2	$T\left(\frac{n}{2^2}\right)$	2^2		$c_2 n + 2^2 c_1$
⋮	⋮	⋮		⋮
i	$T\left(\frac{n}{2^i}\right)$	2^i		$c_2 n + 2^i c_1$
Base case occurs when $\frac{n}{2^i} = 1$ (i.e. $i = \log_2 n$)				
$\log_2 n$	$T(1)$	n	$c_0 + c_0 + \dots + c_0 + c_0$	$2^{\log n} c_0$

The recursion tree has $\lg(n)$ height, each non-leaf node of the tree does $c_2 \frac{n}{2^i} + c_1$ work, each leaf node does c_0 work, and each level has 2^i nodes.

So, the total work is

$$\left(\sum_{i=0}^{\lg(n)-1} 2^i\left(c_1 + c_2\frac{n}{2^i}\right)\right) + c_0 \cdot 2^{\lg(n)} = \left(\sum_{i=0}^{\lg(n)-1} 2^i c_1 + c_2 n\right) + c_0 \cdot (n)$$

$$= c_1 \frac{1-2^{\lg(n)}}{1-2} + c_2 n \lg(n) + c_0 n$$
$$= c_1(n - 1) + c_2 n \lg(n) + c_0 n$$

3. Big-Oof Bounds

Consider the function $f(n)$. Find a recurrence modeling the worst-case runtime of this function and then find a Big-Oh bound for this recurrence.

```
f(n) {
    if (n == 1) {
        return 0
    }

    int result = 0
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            result += j

        }
    }
    return f(n/2) + result + f(n/2)
}
```

a) Find a recurrence $T(n)$ modeling the *worst-case runtime complexity* of $f(n)$

$T(n) = c_0$, if $n = 1$

$T(n) = 2T\left(\frac{n}{2}\right) + c_2\frac{n(n-1)}{2} + c_1$, otherwise

b) Find a Big-Oh bound for your recurrence.

i	Recursive Call	# Nodes	Tree	Sum Work
0	$T(n)$	1	$c_2n^2 + c_1$	$c_2n^2 + c_1$
1	$T\left(\frac{n}{2}\right)$	2	$c_2\left(\frac{n}{2}\right)^2 + c_1$ + $c_2\left(\frac{n}{2}\right)^2 + c_1$	$\frac{1}{2}c_2n^2 + 2c_1$
2	$T\left(\frac{n}{2^2}\right)$	2^2	$c_2\left(\frac{n}{2^2}\right)^2 + c_1$ + $c_2\left(\frac{n}{2^2}\right)^2 + c_1$ + $c_2\left(\frac{n}{2^2}\right)^2 + c_1$ + $c_2\left(\frac{n}{2^2}\right)^2 + c_1$	$\frac{1}{2^2}c_2n^2 + 2^2c_1$
⋮	⋮	⋮		⋮
i	$T\left(\frac{n}{2^i}\right)$	2^i	$c_2\left(\frac{n}{2^i}\right)^2 + c_1$ + $c_2\left(\frac{n}{2^i}\right)^2 + c_1$ + ... + $c_2\left(\frac{n}{2^i}\right)^2 + c_1$ + $c_2\left(\frac{n}{2^i}\right)^2 + c_1$	$\frac{1}{2^i}c_2n^2 + 2^ic_1$
Base case occurs when $\frac{n}{2^i} = 1$ (i.e. $i = \log_2 n$)				
$\log_2 n$	$T(1)$	n	c_0 + c_0 + ... + c_0 + c_0	$2^{\log n}c_0$

Since we only want a Big-Oh, we can actually leave off lower-order terms when doing our analysis, as they won't affect the runtime bounds; so, we can ignore the constants c_1 and c_2 in our analysis.

Note that $\frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \in O\left(n^2\right)$. We can, again, ignore the lower-order term $(\frac{n}{2})$ since we only want a Big-Oh bound.

The recursion tree has $\lg(n)$ height, each non-leaf node of the tree does $\left(\frac{n}{2^i}\right)^2$ work, each leaf node does c_0 work, and each level has 2^i nodes.

So, the total work is:

$$\sum_{i=0}^{\lg(n)-1} 2^i\left(\frac{n}{2^i}\right)^2 + c_0 \cdot 2^{\lg n} = n^2 \sum_{i=0}^{\lg(n)-1} \frac{2^i}{4^i} + c_0 n < n^2 \sum_{i=0}^{\infty} \frac{1}{2^i} + c_0 n = \frac{n^2}{1-\frac{1}{2}} + c_0 n$$

This expression is upper-bounded by n^2 so $T \in \mathcal{O}\left(n^2\right)$.

4. Odds Not in Your Favor

Consider the function $g(n)$. Find a recurrence modeling the worst-case runtime of this function and then find a Big-Oh bound for this recurrence.

```
g(n) {
    if (n <= 1) {
       return 1000
    }
    if (g(n/3) > 5) {
        for (int i = 0; i < n; i++) {
            println("Yay!")
        }
        return 5 * g(n/3)
    } else {
        for (int i = 0; i < n * n; i++) {
            println("Yay!")
        }
        return 4 * g(n/3
    }

```

a) Find a recurrence $T(n)$ modeling the *worst-case runtime complexity* of $g(n)$

$T(n) = c_0$, if $n \leq 1$

$T(n) = 2T\left(\frac{n}{3}\right) + c_2 n + c_1$, otherwise

b) Use your answer in part (a) to find a closed form for $T(n)$

i	Recursive Call	# Nodes	Tree	Sum Work
0	$T(n)$	1		$c_2 n + c_1$
1	$T\left(\frac{n}{3}\right)$	2	+	$\frac{2}{3}c_2 n + 2c_1$
2	$T\left(\frac{n}{3^2}\right)$	2^2	+ + +	$\frac{2^2}{3^2}c_2 n + 2^2 c_1$
⋮	⋮	⋮		⋮
i	$T\left(\frac{n}{3^i}\right)$	2^i	+ + ... + +	$\frac{2^i}{3^i}c_2 n + 2^i c_1$
Base case occurs when $\frac{n}{3^i} = 1$ (i.e. $i = \log_3 n$)				
$\log_3 n$	$T(1)$	n	c_0 + c_0 + ... + c_0 + c_0	$2^{\log_3 n} c_0$

The recursion tree has height $\log_3(n)$, each non-leaf level i has work $\left(\frac{c_2 n}{3^i} + c_1\right)2^i$, and the leaf level has work $c_0 2^{\log_3(n)}$. Putting this together, we have:

$$\sum_{i=0}^{\log_3(n)-1} \left(\left(\frac{c_2 n}{3^i} + c_1\right)2^i\right) + c_0 2^{\log_3(n)}$$

$$= \sum_{i=0}^{\log_3(n)-1} \left(\frac{c_2 n 2^i}{3^i} + c_1 2^i \right) + c_0 2^{\log_3(n)}$$

$$= c_2 n \left(\sum_{i=0}^{\log_3(n)-1} \left(\frac{2}{3}\right)^i \right) + c_1 \left(\sum_{i=0}^{\log_3(n)-1} 2^i \right) + c_0 2^{\log_3(n)}$$

Using the finite geometric series,

$$= c_2 n \left(\frac{1-\left(\frac{2}{3}\right)^{\log_3(n)}}{1-\frac{2}{3}} \right) + c_1 \left(\frac{1-2^{\log_3(n)}}{1-2} \right) + c_0 2^{\log_3(n)} = 3c_2 n \left(1 - \left(\frac{2}{3}\right)^{\log_3(n)}\right) + c_1 \left(2^{\log_3(n)} - 1\right) + c_0 2^{\log_3(n)}$$

$$= 3c_2 n \left(1 - \frac{n^{\log_3(2)}}{n}\right) + c_1 \left(n^{\log_3(2)} - 1\right) + c_0 n^{\log_3(2)}$$

$$= 3c_2 n - 3c_2 n^{\log_3(2)} + c_1 n^{\log_3(2)} - c_1 + c_0 n^{\log_3(2)}$$

$$= 3c_2 n + \left(c_0 + c_1 - 3c_2\right) n^{\log_3(2)} - c_1$$